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Trees with equal domination and tree-free domination numbers

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Abstract

The tree-free domination number $\gamma(G;-\mathscr{T}_k)$, $k \geqslant 2$, of a graph G is the minimum cardinality of a dominating set S in G such that the subgraph $\langle S\rangle$ induced by S contains no tree on k vertices as a (not necessarily induced) subgraph (equivalently, each component of $\langle S\rangle$ has cardinality less than k). When $k=2$, the tree-free domination number is the independent domination number. We obtain a characterization of trees with equal domination and tree-free domination numbers. This generalizes a result of Cockayne et al. (A characterisation of (γ,i)-trees. J. Graph Theory 34(4) (2000) 277–292).

Keywords: Domination number; Independent domination number; Tree-free domination number; Tree

1. Introduction

Domination and its variations in graphs are now well studied (see [3,9,10]). A *dominating set* of a graph G is a set S of vertices of G such that every vertex not in S is adjacent to a vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. A dominating set of G of cardinality $\gamma(G)$ we call a γ-set. An *independent dominating set* of G is a set that is both dominating and independent. The *independent domination number* $i(G)$ is the minimum cardinality of an independent dominating set. An independent dominating set of G of cardinality $i(G)$ we call an i-set.

Haynes et al. [12] initiated the study of $\mathscr{F}$-free domination in graphs. Let $\mathscr{F}$ be a family (possibly infinite) of connected *nontrivial* graphs. A *$\mathscr{F}$-free dominating set* S

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of a graph G is a dominating set of G where the induced subgraph $\langle S\rangle$ contains no graph in $\mathscr{F}$ as a (not necessarily induced) subgraph. The *$\mathscr{F}$-free domination number* $\gamma(G;-\mathscr{F})$ is the minimum cardinality of a $\mathscr{F}$-free dominating set in G. Since $K_1 \notin \mathscr{F}$, the $\mathscr{F}$-free domination number is well-defined for every graph. We refer to a minimum $\mathscr{F}$-free dominating set as a $(\gamma;-\mathscr{F})$-set.

When $\mathscr{F}$ is the family of all cycles, $\gamma(G;-\mathscr{F})$ is the *acyclic domination number* studied by Hedetniemi et al. [13]. When $\mathscr{F}$ consists of a path on at least two vertices, $\gamma(G;-\mathscr{F})$ is the *path-free domination number* studied by Haynes and Henning [11]. When $\mathscr{F}=\{K_{1,k}\}$, $\gamma(G;-\mathscr{F})$ is the *star-free domination number* which is the minimum cardinality of a dominating set of G that induces a graph of maximum degree less than k. The star-free domination number has been studied by Favaron et al. [6] who called it the *k-dependent domination number*. The concept of $\mathscr{F}$-free domination also suggests domination parameters that have not yet been studied. For example, when $\mathscr{F}$ is the family of all homeomorphs of K_5 and $K_{3,3}$, $\gamma(G;-\mathscr{F})$ is the *planar domination number* which is the minimum cardinality of a dominating set of G that induces a planar graph. This method for imposing conditions on the dominating set seems to be rich in applications. Perhaps the most obvious application is network design where there are specialized communication requirements on the processors in the dominating set.

In this paper, we consider $\mathscr{F}=\mathscr{T}_k$ where $\mathscr{T}_k$ is the family of trees on $k \geqslant 2$ vertices. We call $\gamma(G;-\mathscr{T}_k)$ the *tree-free domination number* of G. Thus, $\gamma(G;-\mathscr{T}_k)$ is the minimum cardinality of a dominating set of G that induces a graph each component of which has cardinality less than k. When $k=2$, the tree-free domination number is the independent domination number $i(G)$. When $k \geqslant (n+1)/2$, then, since every γ-set of a graph G without isolated vertices on n vertices has cardinality at most $n/2$, the tree-free domination number is the domination number $\gamma(G)$. Thus, for $k \geqslant 2$, $\gamma(G) \leqslant \gamma(G;-\mathscr{T}_k) \leqslant i(G)$.

In this paper, we present a characterization of trees with equal domination and $\mathscr{T}_k$-free domination number for all integers $k \geqslant 2$. This generalizes the result of Cockayne et al. [4] for $k=2$.

For notation and graph theory terminology we in general follow [3]. Specifically, let $G=(V,E)$ be a graph with vertex set V of order $|V|=n$ and edge set E, and let v be a vertex in V. The open neighborhood of v is $N(v)=\{u \in V \mid uv \in E\}$ and the closed neighborhood of v is $N[v]=\{v\} \cup N(v)$. For a set S of vertices, the open neighborhood of S is defined by $N(S)=\bigcup_{v\in S} N(v)$, and the closed neighborhood of S by $N[S]=N(S)\cup S$. Let $N[v_1,v_2,\ldots,v_k]$ denote the closed neighborhood $N[\{v_1,v_2,\ldots,v_k\}]$ of the set $\{v_1,v_2,\ldots,v_k\}$. A *leaf* is a vertex of degree 1 and its neighbor is called a *support vertex*. The subgraph of G induced by the vertices in S is denoted by $\langle S\rangle$. Following the notation of [3], we denote the maximum order of a component of G by $N(G)$.

For each vertex v in a minimal dominating set S of a graph G, the *private neighborhood* $\mathrm{pn}(v,S)$ of v is given by $N[v]-N[S-\{v\}]$. If $u \in \mathrm{pn}(v,S)$, then either u is isolated in $\langle S\rangle$, in which case $u=v$, or $u \in V-S$ and is adjacent to precisely one vertex of S, namely v.

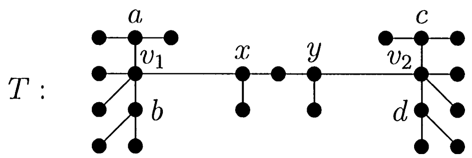

Fig. 1.

2. Main result

For any two graph theoretical parameters λ and μ, we define a graph G to be a (λ, μ)-graph if $\lambda(G) = \mu(G)$. It is well-known that $\gamma(G) \leqslant i(G)$ for all graphs G, and that the class of (γ, i)-graphs is very difficult to characterise. Several classes of (γ, i)-graphs have been found—see, for example, [1,2,5,7,15].

The class of (γ, i)-trees was first characterised in [8] but this characterisation, which involves reducing transformations and forbidden configurations, is rather difficult to use. Recently, Cockayne et al. [4] provided a more elegant characterisation of (γ, i)-trees which is relatively easy to use. Their characterisation is in terms of the sets of vertices of the tree which are contained in all its γ- and i-sets. These sets were characterised by Mynhardt [14] who used an ingenius tree pruning procedure.

Our aim is to generalize the result of Cockayne et al. [4] by providing a characterization of trees with equal domination and $\mathscr{T}_k$-free domination number in terms of the sets of vertices of the tree which are contained in all its γ-sets and $(\gamma; -\mathscr{T}_k)$-sets. For this purpose, we define the sets $\mathscr{A}(G)$, $\mathscr{A}_k(G)$, $\mathscr{A}_{k,0}(G)$, and $\mathscr{A}_{k,1}(G)$ $(k \geqslant 2)$ of a graph G by

- $\mathscr{A}(G) = \{v \in V(G) \,|\, v$ is in every γ-set of $G\}$,
- $\mathscr{A}_k(G) = \{v \in V(G) \,|\, v$ is in every $(\gamma; -\mathscr{T}_k)$-set of $G\}$,
- $\mathscr{A}_{k,0}(G) = \{v \in \mathscr{A}(G) \,|\, v$ is isolated in $\langle \mathscr{A}(G) \rangle$ or, if $k \geqslant 3$, v belongs to a component in $\langle \mathscr{A}(G) \rangle$ of order at most $k-2$ in $\langle \mathscr{A}(G) \rangle\}$,
- $\mathscr{A}_{k,1}(G) = \{v \in \mathscr{A}(G) \,|\, v$ belongs to a component of $\langle \mathscr{A}(G) \rangle$ of order exactly $k-1\}$.

When $k = 2$, the set $\mathscr{A}_k(G)$ is denoted by $\mathscr{A}_i(G)$ in [4,14]. Note that when $k = 2$, $\mathscr{A}_{k,0}(G) = \mathscr{A}_{k,1}(G)$.

For an arbitrary graph G, $\mathscr{A}(G)$ and $\mathscr{A}_k(G)$ are not necessarily related by inclusion. For example, the tree T shown in Fig. 1 satisfies $\gamma(T) = 8$ and $\gamma(T; -\mathscr{T}_4) = 9$. Furthermore, $\mathscr{A}(T) = \{a, b, c, d, v_1, v_2\}$, while $\mathscr{A}_4(T) = \emptyset$. This tree is easily generalized to produce a tree T that satisfies $\gamma(T) = 2k$ and $\gamma(T; -\mathscr{T}_k) = 2k+1$ with $|\mathscr{A}(T)| = 2(k-1)$ and $\mathscr{A}_k(T) = \emptyset$ for any $k \geqslant 2$.

On the other hand, the tree T shown in Fig. 2 satisfies $\mathscr{A}(T) = \{a, b, c, d, v_1, v_2\}$ and $\mathscr{A}_4(T) = \mathscr{A}(T) \cup \{u, v\}$. This tree is easily generalized to produce a tree T that satisfies $|\mathscr{A}(T)| = 2(k-1)$ and $|\mathscr{A}_k(T)| = 2k$ with $\mathscr{A}(T) \subset \mathscr{A}_k(T)$ (and satisfies $\gamma(T) = \gamma(T; -\mathscr{T}_k) = 2k$).

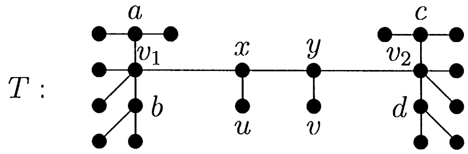

Fig. 2.

If $\gamma(G) = \gamma(G; -\mathscr{T}_k)$, then every $(\gamma; -\mathscr{T}_k)$-set of G is a γ-set of G, and so $\mathscr{A}(G) \subseteq \mathscr{A}_k(G)$. Strict inclusion is possible as illustrated by the tree T of Fig. 2.

For a vertex $v \in V(G) - \mathscr{A}(G)$, we define

$$\begin{aligned} \mathscr{A}^v_{k,1}(G) &= \{w \in \mathscr{A}_{k,1}(G) \mid v \text{ is adjacent to a vertex in the component of } \langle \mathscr{A}(G) \rangle \\ &\quad\ \text{containing } w\}, \text{ and} \\ \mathscr{A}^v_k(G) &= \mathscr{A}(G) - \mathscr{A}_{k,0}(G) - \mathscr{A}^v_{k,1}(G). \end{aligned}$$

We now turn our attention to trees. For any tree T and $v \in V(T) - \mathscr{A}(T)$, we define the subforest $T^*_k(v)$ of T recursively by means of a sequence of subgraphs $T^0_k(v), T^1_k(v), \ldots, T^\ell_k(v)$ of T, where $T^0_k(v) = T$ and, for any $T^j_k(v)$, we define $T^{j+1}_k(v) = T^j_k(v) - N[\mathscr{A}^v_k(T^j_k(v))]$. Since T is finite, there exists an integer ℓ such that either $\mathscr{A}^v_k(T^\ell_k(v)) = \emptyset$ or $v \in \mathscr{A}(T^\ell_k(v))$. Then $T^*_k(v) = T^\ell_k(v)$. We define the subset $P_k(T)$ of $V(T)$ by

$$P_k(T) = \{v \in V(T) - \mathscr{A}(T) \mid v \in \mathscr{A}(T^*_k(v))\}.$$

To illustrate the procedure described above to determine $P_k(T)$ for a tree, let $k=4$ and consider the tree T of Fig. 2. Then $\mathscr{A}(T) = \mathscr{A}_{k,1}(T) = \{a, b, c, d, v_1, v_2\}$ and $\mathscr{A}_{k,0}(T) = \emptyset$. Let u, v, x be the vertices indicated in Fig. 2. Then $\mathscr{A}^v_{k,1}(T) = \emptyset$ and $\mathscr{A}^v_k(T) = \mathscr{A}(T)$. Thus, $T^0_k(v) = T$ and $T^1_k(v) = T - N[\mathscr{A}^v_k(T^0_k(v))] = T - N[\mathscr{A}^v_k(T)] = T - N[\mathscr{A}(T)]$ consists of the isolated vertices u and v. Hence, $\mathscr{A}(T^1_k(v)) = \{u, v\}$. Since $v \in \mathscr{A}(T^1_k(v))$, $T^*_k(v) = T^1_k(v)$ and $v \in \mathscr{A}(T^*_k(v))$, i.e., $v \in P_k(T)$. Similarly, $u \in P_k(T)$. No vertex of $V(T) - \mathscr{A}(T)$, other than u and v, belongs to $P_k(T)$. For example, consider the vertex x. Then $\mathscr{A}^x_{k,1}(T) = \{a, b, v_1\}$ and $\mathscr{A}^x_k(T) = \{c, d, v_2\}$. Thus, $T^0_k(x) = T$ and $T^1_k(x) = T - N[\mathscr{A}^x_k(T^0_k(x))] = T - N[\mathscr{A}^x_k(T)] = T - N[c, d, v_2]$ consists of $\langle N[a, b, v_1] \rangle$ together with the vertices u and v and the edge ux. Hence, $\mathscr{A}(T^1_k(x)) = \{a, b, v, v_1\}$, $\mathscr{A}_{k,0}(T^1_k(x)) = \{v\}$ and $\mathscr{A}^x_{k,1}(T^1_k(x)) = \{a, b, v_1\}$. Thus, $\mathscr{A}^x_k(T^1_k(x)) = \emptyset$. Hence, $T^*_k(x) = T^1_k(x)$ and $\mathscr{A}^x_k(T^*_k(x)) = \emptyset$, i.e., $x \notin P_k(T)$. Consequently, $P_k(T) = \{u, v\}$.

Notice that for the tree T of Fig. 2 and for $k = 4$, $\gamma(T) = \gamma(T; -\mathscr{T}_k)$ and $\mathscr{A}_k(T) = \mathscr{A}(T) \cup P_k(T)$. In particular, each component of $\langle \mathscr{A}(T) \cup P_k(T) \rangle$ has order less than k, i.e., $N(\langle \mathscr{A}(T) \cup P_k(T) \rangle) < k$. We prove that this is true for all trees with equal domination and tree-free domination numbers. We shall prove (see Section 3):

Theorem 1. *For any tree T, $\gamma(T) = \gamma(T; -\mathscr{T}_k)$ if and only if $N(\langle \mathscr{A}(T) \cup P_k(T) \rangle) < k$.*

An immediate corollary now follows.

Corollary 2. *For any tree T, if $\mathscr{A}(T)=\mathscr{A}_{k,0}(T)$, then $\gamma(T)=\gamma(T;-\mathscr{T}_k)$.*

For the special case of Theorem 1, when $k=2$, we have the following result of Cockayne et al. [4].

Corollary 3 (Cockayne et al. [4]). *For any tree T, $\gamma(T)=i(T)$ if and only if $\mathscr{A}(T)\cup P_2(T)$ is an independent set.*

3. Proof of Theorem 1

3.1. Proof of the necessity

The following result shows that if we remove the vertices of a component of $\langle\mathscr{A}(T)\rangle$ of order exactly $k-1$ and their neighborhoods from a tree T with $\gamma(T)=\gamma(T;-\mathscr{T}_k)$, then the resulting forest F satisfies $\gamma(F)=\gamma(F;-\mathscr{T}_k)$.

Lemma 4. *If T is a tree with $\gamma(T)=\gamma(T;-\mathscr{T}_k)$, $S\subseteq\mathscr{A}_{k,1}(T)$ such that each component of $\langle S\rangle$ has order $k-1$, and $F=T-N[S]$, then $\gamma(F)=\gamma(F;-\mathscr{T}_k)=\gamma(T)-|S|$.*

Proof. If $V(T)=N[S]$, the result holds vacuously. Suppose, then, that $V(T)-N[S]\neq\emptyset$. Let X be a $(\gamma;-\mathscr{T}_k)$-set of T. Since $\gamma(T)=\gamma(T;-\mathscr{T}_k)$, X is also a γ-set of T whence $\mathscr{A}(T)\subseteq X$ and $\mathscr{A}(T)=\mathscr{A}_{k,0}(T)\cup\mathscr{A}_{k,1}(T)$. In particular, $S\subseteq X$. Since X contains no vertex in $N[S]-S$, $X-S$ is a $\mathscr{T}_k$-free dominating set of F, and so $\gamma(F;-\mathscr{T}_k)\leqslant|X|-|S|=\gamma(T)-|S|$. On the other hand, if Y is a $(\gamma;-\mathscr{T}_k)$-set of F, then $Y\cup S$ is a dominating set of T, and so $\gamma(T)\leqslant|Y|+|S|=\gamma(F;-\mathscr{T}_k)+|S|$. Consequently, $\gamma(F;-\mathscr{T}_k)=\gamma(T)-|S|$. Furthermore, if Z is a γ-set of F, $Z\cup S$ is a dominating set of T, and so $\gamma(T)\leqslant|Z|+|S|=\gamma(F)+|S|$. Hence, $\gamma(T)-|S|\leqslant\gamma(F)\leqslant\gamma(F;-\mathscr{T}_k)=\gamma(T)-|S|$. We therefore have equality throughout this chain. In particular, $\gamma(F)=\gamma(F;-\mathscr{T}_k)=\gamma(T)-|S|$. $\square$

As observed earlier, if $\gamma(T)=\gamma(T;-\mathscr{T}_k)$, then every $(\gamma;-\mathscr{T}_k)$-set of T is a γ-set of T, and so $\mathscr{A}(T)\subseteq\mathscr{A}_k(T)$. We state this as a lemma.

Lemma 5. *If T is a tree with $\gamma(T)=\gamma(T;-\mathscr{T}_k)$, then $\mathscr{A}(T)\subseteq\mathscr{A}_k(T)$.*

Lemma 6. *If T is a tree with $\gamma(T)=\gamma(T;-\mathscr{T}_k)$, then $P_k(T)\subseteq\mathscr{A}_k(T)-\mathscr{A}(T)$.*

Proof. Suppose $v\in P_k(T)$. By definition, $v\notin\mathscr{A}(T)$. Let X^0 be a $(\gamma;-\mathscr{T}_k)$-set of T. Since $\gamma(T)=\gamma(T;-\mathscr{T}_k)$, X^0 is a γ-set of T and therefore $\mathscr{A}(T)\subseteq X^0$. Furthermore, $\mathscr{A}(T)=\mathscr{A}_{k,0}(T)\cup\mathscr{A}_{k,1}(T)$. Let $S^0=\mathscr{A}_k^v(T)$. Then, $v\notin S^0$ and $S^0=\mathscr{A}_{k,1}(T)-\mathscr{A}_{k,1}^v(T)$. Thus, each component of $\langle S^0\rangle$ has order $k-1$. By Lemma 4, $T_k^1(v)=T-N[S^0]$ satisfies $\gamma(T_k^1(v))=\gamma(T_k^1(v);-\mathscr{T}_k)=\gamma(T)-|S^0|$. Let $X^1=X^0-S^0$. Then X^1 is a $\mathscr{T}_k$-free dominating set of $T_k^1(v)$ of cardinality $\gamma(T) - |S^0| = \gamma(T_k^1(v))$. Hence, X^1 is a γ-set of $T_k^1(v)$ and therefore $\mathscr{A}(T_k^1(v)) \subseteq X^1$. We repeat this process until $T_k^*(v) = T_k^\ell(v)$ is obtained. Note that, for $i = 0, \ldots, \ell - 1$, $S^i = \mathscr{A}_k^v(T_k^i(v))$, $X^{i+1} = X^i - S^i$ and $\mathscr{A}(T_k^{i+1}(v)) \subseteq X^{i+1}$. By definition, $v \in \mathscr{A}(T_k^*(v))$, and so $v \in \mathscr{A}(T_k^\ell(v)) \subseteq X^\ell = X^0 - S^0 - S^1 - \cdots - S^{\ell-1}$. Since $v \notin S^i$ for $i = 0, \ldots, \ell - 1$, $v \in X^0$. However, X^0 was an arbitrary $(\gamma; -\mathscr{T}_k)$-set of T. Hence, $v \in \mathscr{A}_k(T)$. Consequently, $P_k(T) \subseteq \mathscr{A}_k(T) - \mathscr{A}(T)$. $\square$

If T is a tree with $\gamma(T) = \gamma(T; -\mathscr{T}_k)$, then, by Lemmas 5 and 6, $\mathscr{A}(T) \cup P_k(T) \subseteq \mathscr{A}_k(T)$. Since each component of $\langle \mathscr{A}_k(T) \rangle$ has order less than k, so too does each component of $\langle \mathscr{A}(T) \cup P_k(T) \rangle$. This proves the necessity of Theorem 1.

3.2. Proof of the sufficiency

In order to prove the sufficiency, we shall need some results of Cockayne et al. [4]. Let v be a vertex in a rooted tree T, and let $x \in N(v)$. For notational convenience, we may assume $x \in C(v)$. Cockayne et al. [14] defined x to be *v-noble* if there exists a γ-set S of T_x such that $\mathrm{pn}(s, S) = \{x\}$ for some $s \in S$. Cockayne et al. [14] provided the following characterisations of v-noble vertices.

Theorem 7 (Cockayne et al. [4]). *Let v be a vertex of a rooted tree T and let $x \in C(v)$. Then x is v-noble if and only if for each $y \in N(x) - \{v\}$, y does not belong to any γ-set of T_y.*

Cockayne et al. [4] also characterised the set $\mathscr{A}(T)$ in terms of v-noble vertices.

Theorem 8 (Cockayne et al. [4]). *A vertex v of a tree T is in $\mathscr{A}(T)$ if and if $N(v)$ contains at least two v-noble vertices.*

We will also need the following results from [4] and [14].

Lemma 9 (Mynhardt [14]). *If T is a tree with $\mathscr{A}(T) = \emptyset$, then $\mathscr{A}_2(T) = \emptyset$.*

Lemma 10 (Cockayne et al. [4]). *If T is a tree with $\mathscr{A}(T) = \emptyset$, then $\gamma(T) = i(T)$.*

We now return to the proof of the sufficiency. Let X be a γ-set of a tree T. Suppose that u and v are adjacent vertices in X and that $v \notin \mathscr{A}(T)$. We may assume that T is rooted at u or v. For each vertex w of T, let T_w denote the subtree of T consisting of w and the descendants of w. For each vertex $z \in \mathrm{pn}(v; X)$, let $X_z = X \cap V(T_z)$. The proofs of the following three claims are similar to those found in [4], but we include them for completeness.

Claim 1. *For each $z \in \mathrm{pn}(v; X)$, X_z is a γ-set of $T_z - z$.*

Proof. Since $z \notin X$, the set X_z is a dominating set of $T_z - z$, and so $|X_z| \geqslant \gamma(T_z - z)$. If $|X_z| > \gamma(T_z - z)$, then let S_z be a γ-set of $T_z - z$. Then, $(X - X_z) \cup S_z$ is a dominating set of T of cardinality less than $|X|$, a contradiction. Hence, X_z is a γ-set of $T_z - z$. □

Claim 2. *There exists a v-noble vertex $x \in \mathrm{pn}(v, X)$.*

Proof. Suppose, to the contrary, that no vertex in $\mathrm{pn}(v,X)$ is v-noble. Then, by Theorem 7, each $z \in \mathrm{pn}(v,X)$ has a child z' that belongs to some γ-set of $T_{z'}$, say S_z. Since $z \notin X$, $X_{z'}$ is a dominating set of $T_{z'}$, and so $|X_{z'}| \geqslant \gamma(T_{z'}) = |S_z|$. Let

$$X^* = \left(X - \{v\} - \bigcup_{z \in \mathrm{pn}(v,X)} X_{z'} \right) \cup \left(\bigcup_{z \in \mathrm{pn}(v,X)} S_z \right).$$

Then, X^* is a dominating set of T with $|X^*| \leqslant |X| - 1 = \gamma(T) - 1$, which is impossible. Hence, at least one vertex in $\mathrm{pn}(v,X)$ is v-noble. □

Claim 3. *For each $z \in \mathrm{pn}(v;X) - \{x\}$, $\gamma(T_z) \leqslant \gamma(T_z - z) = |X_z|$.*

Proof. Suppose, to the contrary, that $\gamma(T_z) > \gamma(T_z - z)$. Let S_z be a γ-set of $T_z - z$. Since S_z is not a dominating set of T_z, z is not adjacent to any vertex of S_z. Thus, $S_z \cup \{z\}$ is a dominating set, and hence a γ-set, of T_z with $\mathrm{pn}(z, S_z) = \{z\}$. But then z is a v-noble vertex by definition. Hence, $N(v)$ contains at least two v-noble vertices, namely, x and z. Thus, by Theorem 8, $v \in \mathscr{A}(T)$, a contradiction. □

For a subset U of $\mathscr{A}(T)$, let C_U be the component(s) in $\langle \mathscr{A}(T) \rangle$ containing all the vertices of U, and let $V_U = V(C_U)$. Further, let $N_U = N[V_U] - V_U$.

Lemma 11. If T is a tree, $U \subseteq \mathscr{A}(T)$, and $T' = T - N[V_U]$, then $\mathscr{A}(T) - V_U \subseteq \mathscr{A}(T')$.

Proof. Necessarily, each vertex of $\mathscr{A}(T)$ has a neighbor not in $\mathscr{A}(T)$. Suppose that there exists a γ-set X' of T' such that $v \notin X'$ where $v \in \mathscr{A}(T) - V_U$. Let $X = X' \cup V_U$. Then X dominates T and $v \notin X$, and so X cannot be a γ-set of T. Let Y be a γ-set of T. Then $|Y| < |X|$. If $Y \cap N_U = \emptyset$, then, since no vertex in V_U is adjacent (in T) to a vertex of T', $Y' = Y - V_U$ dominates T'. Hence, $\gamma(T') \leqslant |Y'| = |Y| - |V_U| < |X| - |V_U| = |X'|$, a contradiction. Hence there exists a vertex $w \in Y \cap N_U$. Proceeding now exactly as in the proof of Theorem 3 in [14], we can show that $w \in \mathscr{A}(T)$. This, however, produces a contradiction. Hence every γ-set of T' contains all the vertices in $\mathscr{A}(T) - V_U$. □

We are now in a position to present the following lemmas.

Lemma 12. *If T is a tree for which $\mathscr{A}(T) = \emptyset$, then $P_k(T) = \mathscr{A}_k(T) = \emptyset$ and $\gamma(T) = \gamma(T; -\mathscr{T}_k)$.*

Proof. Since $\mathscr{A}(T) = \emptyset$, Lemma 10 implies that $\gamma(T) = i(T)$. However, $\gamma(G) \leqslant \gamma(G; -\mathscr{T}_k) \leqslant i(G)$ for all graphs G. Consequently, $\gamma(T) = \gamma(T; -\mathscr{T}_k) = i(T)$. Hence every i-set of T is a $(\gamma; -\mathscr{T}_k)$-set of T. Thus, $\mathscr{A}_k(T) = \emptyset$, for otherwise if $v \in \mathscr{A}_k(T)$, then v must belong to every i-set of T, i.e., $v \in \mathscr{A}_2(T)$. However, by Lemma 9, $\mathscr{A}_2(T) = \emptyset$ and we have a contradiction. Hence if $\mathscr{A}(T) = \emptyset$, then $\mathscr{A}_k(T) = \emptyset$. Furthermore, since $\mathscr{A}(T) = \emptyset$, $\mathscr{A}_k^v(T) = \emptyset$ for every vertex $v \in V(T) - \mathscr{A}(T)$, whence $T_k^*(v) = T_k^0(v) = T$ and $\mathscr{A}_k^v(T_v^*) = \emptyset$, i.e., $v \notin P_k(T)$. Consequently, $P_k(T) = \emptyset$. $\square$

Lemma 13. *If T is a tree for which $\mathscr{A}(T) = \mathscr{A}_{k,0}(T)$, then $P_k(T) = \emptyset$.*

Proof. Since $\mathscr{A}(T) = \mathscr{A}_{k,0}(T)$, $\mathscr{A}_k^v(T) = \emptyset$ for every vertex $v \in V(T) - \mathscr{A}(T)$, whence $T_k^*(v) = T_k^0(v) = T$ and $\mathscr{A}_k^v(T_k^*(v)) = \emptyset$, i.e., $v \notin P_k(T)$. Consequently, $P_k(T) = \emptyset$. $\square$

Lemma 14. *If T is a tree and $N(\langle \mathscr{A}(T) \cup P_k(T) \rangle) < k$, then $\gamma(T) = \gamma(T; -\mathscr{T}_k)$.*

Proof. We proceed by induction on $|\mathscr{A}(T) \cup P_k(T)|$. If $\mathscr{A}(T) = \emptyset$, then, by Lemma 12, $P_k(T) = \emptyset$ and $\gamma(T) = \gamma(T; -\mathscr{T}_k)$. Thus the lemma holds when $\mathscr{A}(T) = \emptyset$. In particular, the lemma holds when $|\mathscr{A}(T) \cup P_k(T)| = 0$.

Suppose the lemma holds for all trees H for which $N(\langle \mathscr{A}(H) \cup P_k(H) \rangle) < k$ and with $|\mathscr{A}(H) \cup P_k(H)| < n$, where $n \geqslant 1$, and let T be a tree for which $N(\langle \mathscr{A}(T) \cup P_k(T) \rangle) < k$ and with $|\mathscr{A}(T) \cup P_k(T)| = n$. Then $\mathscr{A}(T) \neq \emptyset$, for otherwise $|\mathscr{A}(T) \cup P_k(T)| = 0$ as observed earlier. Furthermore, $\mathscr{A}(T) = \mathscr{A}_{k,0}(T) \cup \mathscr{A}_{k,1}(T)$. We consider two possibilities.

Case 1: $\mathscr{A}_{k,1}(T) \neq \emptyset$.

Since $N(\langle \mathscr{A}(T) \cup P_k(T) \rangle) < k$, no vertex of $P_k(T) - \mathscr{A}(T)$ is adjacent to any vertex of $\mathscr{A}_{k,1}(T)$. Hence, $\mathscr{A}_{k,1}^v(T) = \emptyset$ for all $v \in P_k(T)$. In particular, $P_k(T)$ contains no vertex in $N[\mathscr{A}_{k,1}(T)]$.

Let $U = \mathscr{A}_{k,1}(T)$ and let $F = T - N[V_U]$. Then, by Lemma 11, $\mathscr{A}(T) - V_U \subseteq \mathscr{A}(F)$. Moreover, if $w \in \mathscr{A}(F) - \mathscr{A}(T)$, then $T_k^*(w) = T_{k,1}(w) = T - N[\mathscr{A}_k^w(T)] = T - N[\mathscr{A}_{k,1}(T)] = T - N[U] = F$, and so $w \in \mathscr{A}(T_k^*(w))$ and therefore $w \in P_k(T)$. Thus, $\mathscr{A}(F) \subseteq \mathscr{A}(T) \cup P_k(T)$. If $w \in P_k(F)$, then $w \in \mathscr{A}(T_k^*(w))$, and so $w \in P_k(T)$. Thus, $\mathscr{A}(F) \cup P_k(F) \subset \mathscr{A}(T) \cup P_k(T)$. Hence, $N(\langle \mathscr{A}(F) \cup P_k(F) \rangle) < k$ and $|\mathscr{A}(F) \cup P_k(F)| < n$. Applying the inductive hypothesis to each component F_i of F, we obtain $\gamma(F_i) = \gamma(F_i; -\mathscr{T}_k)$ and therefore $\gamma(F) = \gamma(F; -\mathscr{T}_k)$.

Let S' be a $(\gamma; -\mathscr{T}_k)$-set of F. Then $S = S' \cup V_U$ is a $\mathscr{T}_k$-free dominating set of T. Suppose $|S| > \gamma(T)$. Let X be a γ-set of T with a minimum number of vertices that belong to a component of $\langle X \rangle$ of order k or more. If $X \cap N[V_U] = V_U$, then $X - V_U$ is a dominating set of F with $\gamma(F) \leqslant |X - V_U| < |S - V_U| = |S'| = \gamma(F)$, which is impossible. Hence, X must contain a vertex $v \in N[V_U] - V_U$. Note that v belongs to a component of order at least k in $\langle X \rangle$. Let u be a vertex of V_U that is adjacent to v. We may assume that T is rooted at u.

Since $V_u \subset X$ and V_u dominates $N[V_u]$, it follows that $\mathrm{pn}(v; X) \subseteq V(F)$. For each vertex $z \in \mathrm{pn}(v; X)$, let $X_z = X \cap V(T_z)$.

By Claim 2, there exists a vertex $x \in \mathrm{pn}(v;X)$ such that T_x contains a γ-set W, where $\mathrm{pn}(w;W)=\{x\}$ for some $w \in W$. For each $z \in \mathrm{pn}(v;X)-\{x\}$, T_z is a component of F and therefore $\gamma(T_z)=\gamma(T_z;-\mathscr{T}_k)$ by the inductive hypothesis. Hence, by Claim 3, $\gamma(T_z;-\mathscr{T}_k) \leqslant \gamma(T_z - z)=|X_z|$. For each $z \in \mathrm{pn}(v;X)-\{x\}$, let Y_z be a $(\gamma;-\mathscr{T}_k)$-set of T_z. Then $|Y_z|=\gamma(T_z;-\mathscr{T}_k) \leqslant |X_z|$. Let $Y'=\cup Y_z$ and let $X'=\cup X_z$ where the unions are taken over all vertices $z \in \mathrm{pn}(v;X)-\{x\}$. Further, let $Y''=(X-X')\cup Y'$ and let $Y=(Y''-\{v\})\cup\{x\}$. Then Y is a dominating set of T and by Claim 1 and the fact that $|Y_z| \leqslant |X_z|$ for all $z \in \mathrm{pn}(v;X)-\{x\}$, $\gamma(T) \leqslant |Y| \leqslant |X|=\gamma(T)$. Consequently, Y is a γ-set of T. However, $x \in \mathrm{pn}(v;X)$ and $v \notin Y$, and so x is isolated in $\langle Y\rangle$. Furthermore, since $N(\langle Y_z\rangle)<k$, it follows that Y is a γ-set of T that contains fewer vertices that belong to components of $\langle Y\rangle$ of order k or more than does X, contradicting our choice of X. We deduce, therefore, that our supposition that $|S|>\gamma(T)$ is false. Consequently, $|S|=\gamma(T)$, whence $\gamma(T)=\gamma(T;-\mathscr{T}_k)$.

Case 2: $\mathscr{A}(T)=\mathscr{A}_{k,0}(T)$.

Let $U=\mathscr{A}_{k,0}(T)$ and let $F=T-N[U]$. Before proceeding further, we prove the following claim.

Claim 4. $\gamma(F)=\gamma(F;-\mathscr{T}_k)$.

Proof. Let I_U be the set of isolated vertices of F and let H be a component of $F-I_U$. Suppose $v \in \mathscr{A}(H)-\mathscr{A}(T)$. Let X be a γ-set of H. Since $v \in \mathscr{A}(H)$, $v \in X$. We may assume that H is rooted at v. Using the notation introduced in Case 1, we may show that Claims 1, 2 and 3 once again hold since $v \notin \mathscr{A}(T)$. For each $z \in \mathrm{pn}(v;X)-\{x\}$, let Y_z be a γ-set of T_z. Then, with the set Y defined as in Case 1, Y is a dominating set of H. By Claim 3, $|Y_z| \leqslant |X_z|$ for all $z \in \mathrm{pn}(v;X)-\{x\}$, and so $\gamma(H) \leqslant |Y| \leqslant |X|=\gamma(H)$. Consequently, Y is a γ-set of H. However, $v \notin Y$. This contradicts the fact that $v \in \mathscr{A}(H)$. Hence every component H of $F-I_U$ satisfies $\mathscr{A}(H)=\mathscr{A}(T)\cap V(H)$. In particular, $\mathscr{A}(H) \subseteq \mathscr{A}_{k,0}(T)$ and therefore $\mathscr{A}(H)=\mathscr{A}_{k,0}(H)$. Consequently, $P_k(H)=\emptyset$ by Lemma 13. Hence, $N(\langle\mathscr{A}(H)\cup P_k(H)\rangle)<k$ and $|\mathscr{A}(H)\cup P_k(H)|<n$. Thus, we may apply the inductive hypothesis to H to obtain $\gamma(H)=\gamma(H;-\mathscr{T}_k)$. It follows that $\gamma(F-I_U)=\gamma(F-I_U;-\mathscr{T}_k)$ and therefore that $\gamma(F)=\gamma(F;-\mathscr{T}_k)$. □

By Claim 4, $\gamma(F)=\gamma(F;-\mathscr{T}_k)$. Let S' be a $(\gamma;-\mathscr{T}_k)$-set of F. Then $S=S'\cup U$ is a $\mathscr{T}_k$-free dominating set of T. Suppose $|S|>\gamma(T)$. Let X be a γ-set of T that contains as few vertices of N_U as possible. If $X\cap N[U]=U$, then $X-U$ is a dominating set of F with $\gamma(F) \leqslant |X-U|<|S-U|=|S'|=\gamma(F)$, which is impossible. Hence, X must contain a vertex $v \in N_U$. Let u be a vertex of U that is adjacent to v. We may assume that T is rooted at u. Since $U \subset X$ and U dominates $N[U]$, it follows that $\mathrm{pn}(v;X) \subseteq V(F)$. Using the notation introduced in Case 1, we may show that Claims 1, 2 and 3 once again hold. For each $z \in \mathrm{pn}(v;X)-\{x\}$, let Y_z be a γ-set of T_z. Then, with the set Y defined as in Case 1, Y is a dominating set of T. By Claim 3, $|Y_z| \leqslant |X_z|$ for all $z \in \mathrm{pn}(v;X)-\{x\}$, and so $\gamma(T) \leqslant |Y| \leqslant |X|=\gamma(T)$. Consequently, Y is a γ-set of T. However, Y contains fewer vertices of N_U than does X, contradicting our choice of X. We deduce, therefore, that our supposition that $|S| > \gamma(T)$ is false. Consequently, $|S| = \gamma(T)$, whence $\gamma(T) = \gamma(T; -\mathscr{T}_k)$. □

Corollary 2 follows from Lemmas 13 and 14.

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